§4.4. Change variables in $\iint_D f\,dA$

#2. $x = u^2 - v^2$, $y = u^2 + v^2$. Find the Jacobian of transformation

sol: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 4uv + 4uv$
$= \underline{8uv}$ #

#4. $x = \alpha \sin\beta$, $y = \alpha\cos\beta$. Find the Jacobian of transformation

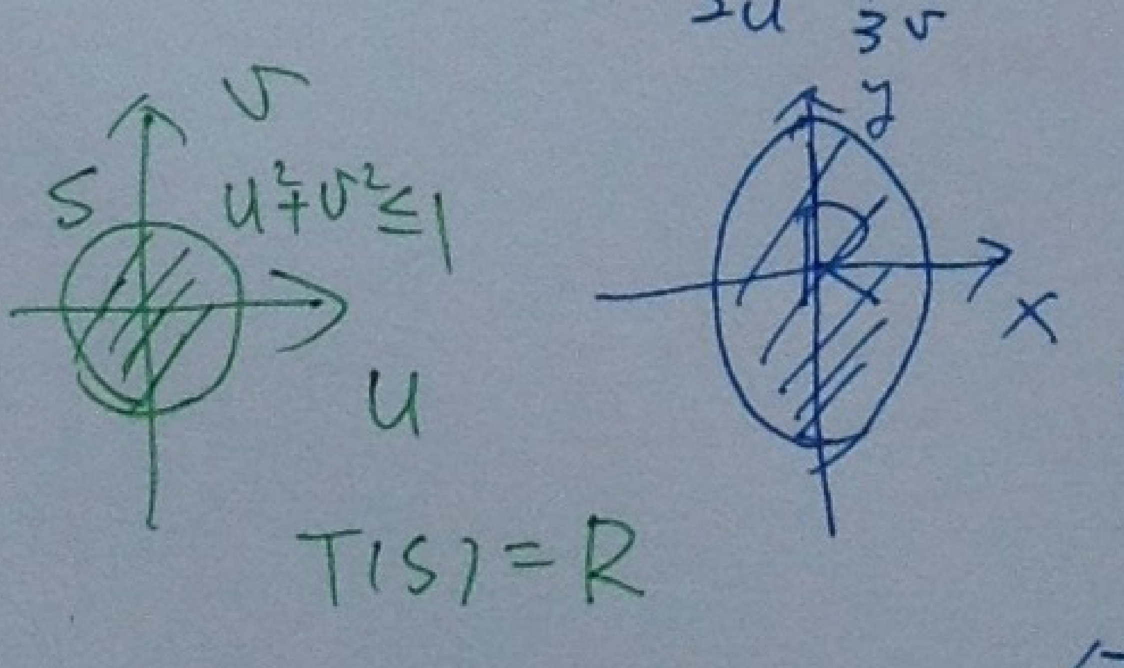
sol: $\dfrac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{vmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{vmatrix} = \begin{vmatrix} \sin\beta & \alpha\cos\beta \\ \cos\beta & -\alpha\sin\beta \end{vmatrix} = -\alpha\sin^2\beta - \alpha\cos^2\beta$
$= \underline{-\alpha}$ #

#13. Use the given transform to evaluate the integral

$\iint_R x^2\,dA$, R: bdd by ellipse $9x^2 + 4y^2 = 36$;

$x = 2u$, $y = 3v$

sol: $(u,v) \xrightarrow{T} (x,y)$, $x = 2u$, $y = 3v$



$\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

$\dfrac{(2u)^2}{4} + \dfrac{(3v)^2}{9} = 1$

$\Longleftrightarrow u^2 + v^2 = 1$

$\iint_R x^2\,dA$

$= \iint_S 4u^2\,|J|\,du\,dv$

$J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$

$\iint_R x^2\,dA = \iint_S 24u^2\,\underline{du\,dv}$ $\underset{\text{polar coordin}}{=}$ $\int_0^{2\pi}\int_0^1 24r^2\cos^2\theta\ \underline{r\,dr\,d\theta}$

$u = r\cos\theta$
$v = r\sin\theta$

$= 6r^4\Big|_0^1 \cdot 2\pi \times \frac{1}{2}$

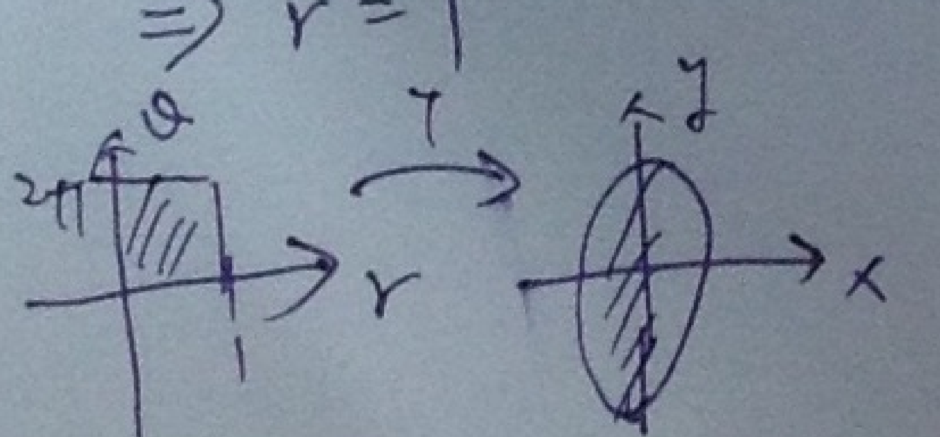
$= \underline{6\pi}$ #

(註) 若沒要求使用已給的變換

可直接令 $T\begin{cases} x = 2r\cos\theta \\ y = 3r\sin\theta \end{cases}$

$\Rightarrow r = 1$

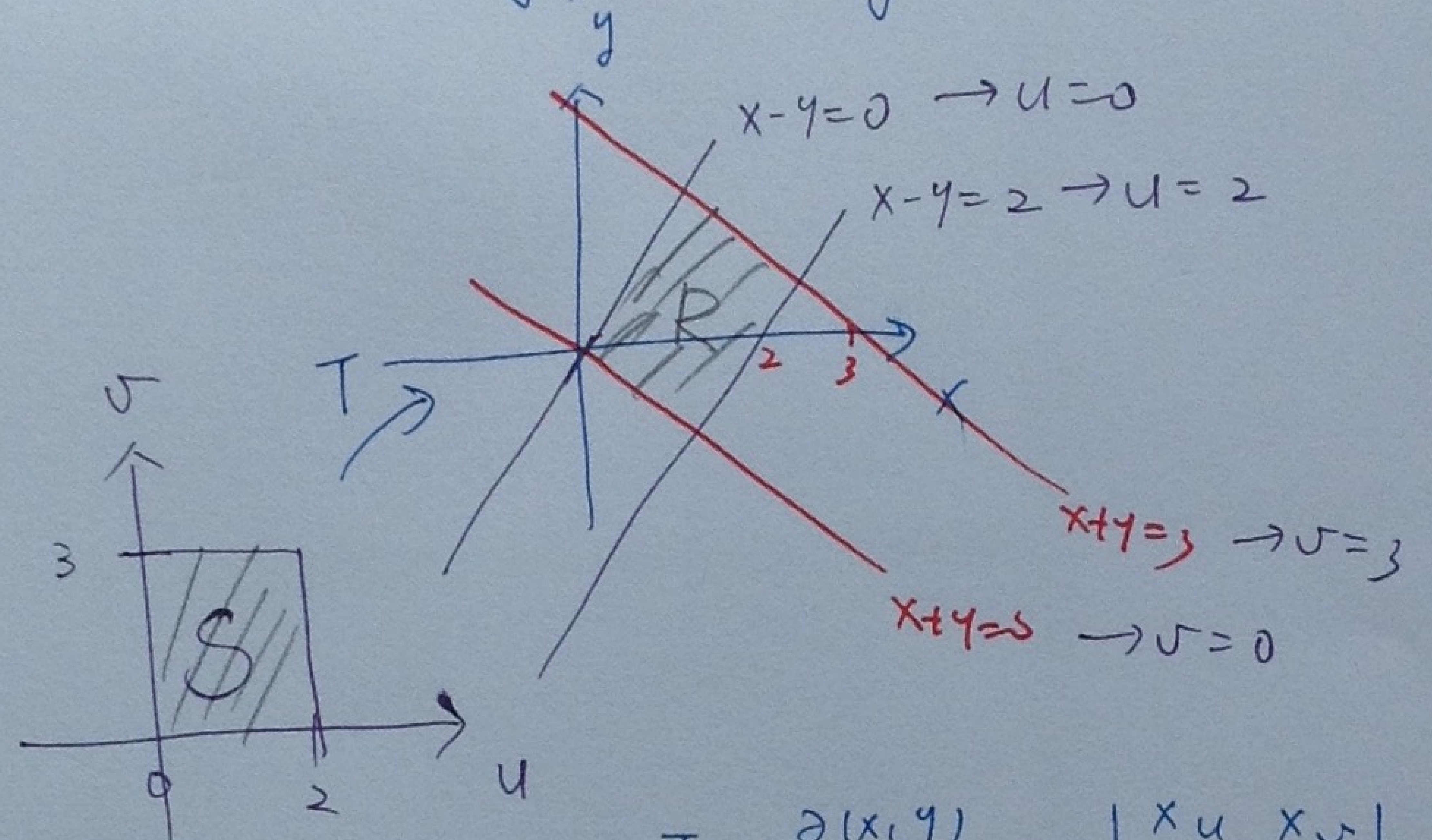
亦可做出 6π #



§4.4. Change variables in $\iint_D f \, dA$

#20. $\iint_R (x+y) e^{x^2-y^2} \, dA$, R: rectangles enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$

sol: Let $u = x-y$, $v = x+y$



$$\begin{cases} u = x-y \\ v = x+y \end{cases} \Rightarrow \begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\iint_R (x+y) e^{x^2-y^2} \, \underline{dA} = dx\,dy = |J| \, du\,dv$$

$$= \iint_S v \, e^{uv} \, \underline{\tfrac{1}{2} du\,dv}$$

$$= \frac{1}{2} \int_0^3 \left(\int_{u=0}^{2} e^{uv} v \, du \right) dv = \frac{1}{2} \int_0^3 \left(e^{uv} \Big|_{u=0}^{u=2} \right) dv$$

$$= \frac{1}{2} \int_0^3 (e^{2v} - 1) \, dv$$

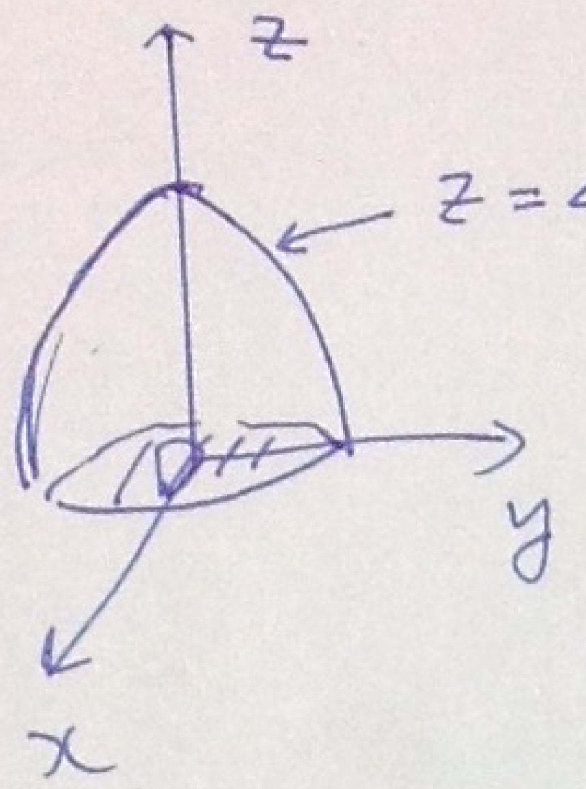
$$= \frac{1}{2} \cdot \frac{1}{2} e^{2v} \Big|_{v=0}^{3} - \frac{3}{2}$$

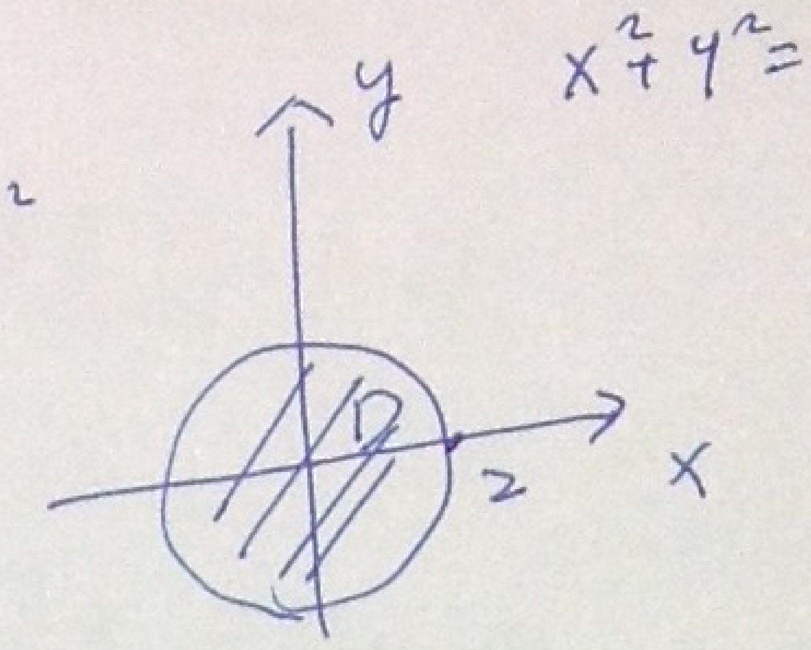
$$= \frac{1}{4}(e^6 - 1) - \frac{3}{2} = \underline{\frac{1}{4}(e^6 - 7)} \; \#$$

§4.4 II. Surface Area

#6. The part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.

sol:





$D: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{cases}$

$f_x = -2x \,,\quad f_y = -2y$

$$\text{Area}(S) = \iint_D \sqrt{1+4x^2+4y^2}\, dA$$

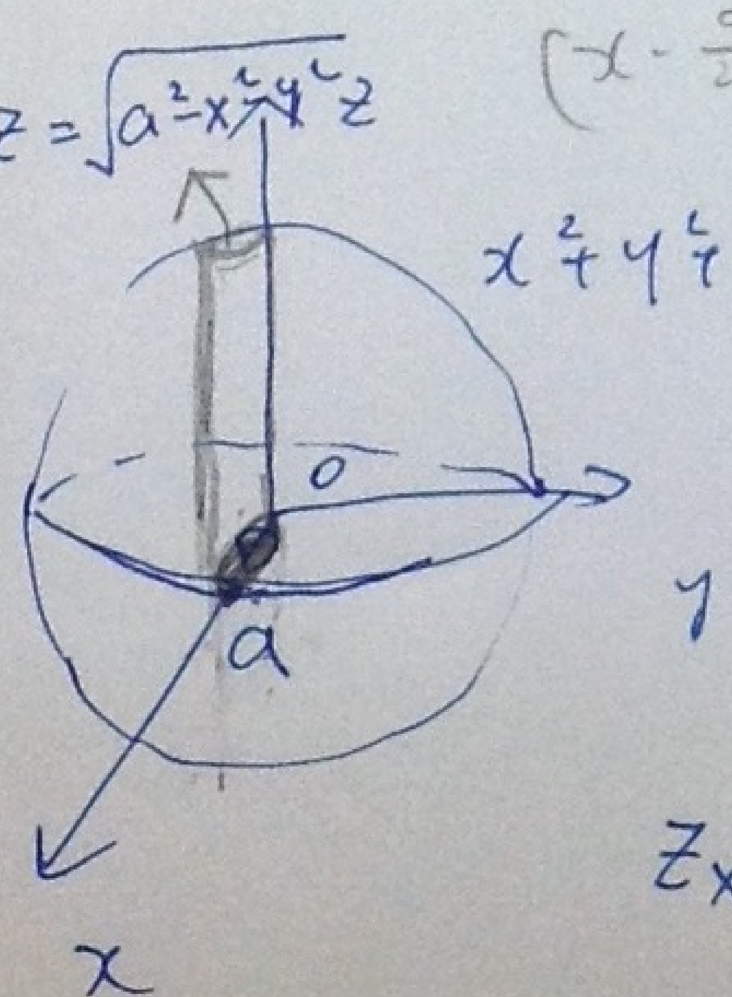
$$= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\; r\,dr\,d\theta = 2\pi \cdot \int_0^2 (1+4r^2)^{\frac{1}{2}}\, d(r^2 4+1)\cdot \frac{1}{8}$$

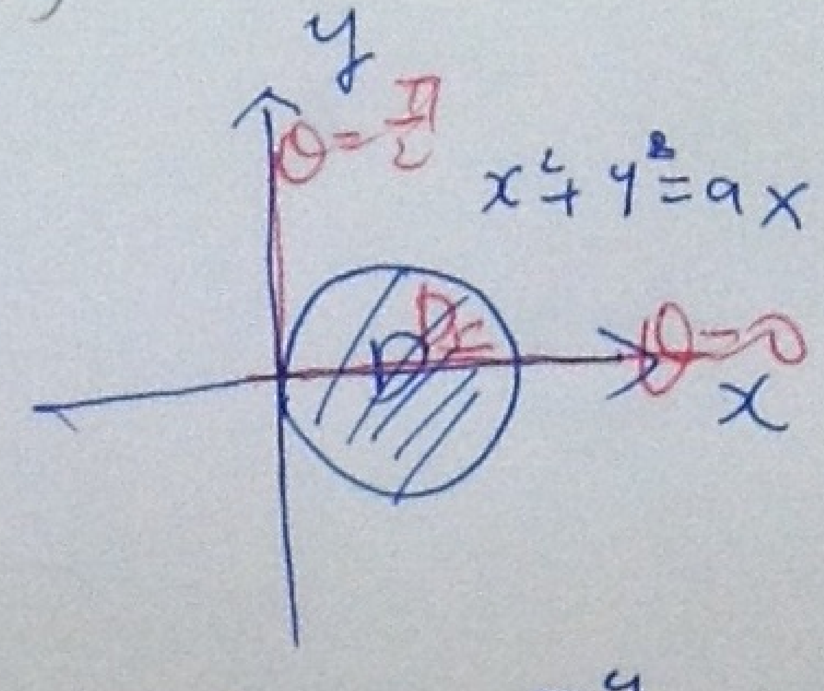
$$= 2\pi\, (1+4r^2)^{\frac{3}{2}}\Big|_{r=0}^{2} \frac{2}{3}\cdot\frac{1}{8} = \underline{\frac{\pi}{6}\left(17^{\frac{3}{2}}-1\right)}$$ ※

#11. The part of the sphere $x^2+y^2+z^2=a^2$ that lies within the cylinder $x^2+y^2=ax$ and above the xy-plane., $a>0$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

sol:





$r^2 = ar\cos\theta$

$\boxed{r = a\cos\theta}$

$z_x = \dfrac{-x}{\sqrt{a^2-x^2-y^2}} \qquad z_y = \dfrac{-y}{\sqrt{a^2-x^2-y^2}}$

$D_上: \begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le a\cos\theta \end{cases}$

$$\text{Area}(S) = \iint_D \sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\; dA$$

$$= \iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\, dA$$ 对称 $y=0$ 而 integrand 对 y 是 even

$$= 2\iint_{D_上} \frac{a}{\sqrt{a^2-x^2-y^2}}\, dA = 2\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \underbrace{\frac{a}{\sqrt{a^2-r^2}}}_{=\,a(a^2-r^2)^{-\frac{1}{2}}}\; \underbrace{r\,dr}_{\frac{1}{2}dr^2 = -\frac{1}{2}d(a^2-r^2)}\,d\theta$$

$$= -a\int_0^{\frac{\pi}{2}} 2(a^2-r^2)^{\frac{1}{2}}\Big|_{r=0}^{r=a\cos\theta} d\theta = 2a^2\int_0^{\frac{\pi}{2}}(1-\sin\theta)\,d\theta = \underline{a^2(\pi-1)}$$ ※